Let $\vec{A}, \vec{B}, \vec{C}$ be three vectors show that

1) $\frac{d}{dt}(\vec{A} \times \vec{B}) = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}$

2) $\frac{d}{dt}(\phi \vec{A}) = \frac{d\phi}{dt}\vec{A} + \phi \frac{d\vec{A}}{dt}$

~~3) $\frac{d}{dt}(\vec{A} \cdot (\vec{B} \times \vec{C})) = \vec{A} \cdot \vec{B} \times \frac{d\vec{C}}{dt} + \vec{A} \cdot \frac{d\vec{B}}{dt} \times \vec{C} + \frac{d\vec{A}}{dt} \cdot \vec{B} \times \vec{C}$~~

Sol: 1) $\frac{d}{dt}(\vec{A} \times \vec{B}) = \lim_{\Delta t \to 0} \frac{(\vec{A}+\Delta\vec{A}) \times (\vec{B}+\Delta\vec{B}) - \vec{A} \times \vec{B}}{\Delta t}$

$$= \lim_{\Delta t \to 0} \frac{\Delta\vec{A} \times \vec{B} + \vec{A} \times \Delta\vec{B} + \Delta\vec{A} \times \Delta\vec{B}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\Delta\vec{A}}{\Delta t} \times \vec{B} + \lim_{\Delta t \to 0} \vec{A} \times \frac{\Delta\vec{B}}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta\vec{A}}{\Delta t} \times \Delta\vec{B}$$

$$= \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt} + \lim_{\Delta t \to 0} \frac{\Delta\vec{A}}{\Delta t} \times \frac{\Delta\vec{B}}{\Delta t} \Delta t$$

$$= \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt} \times \frac{d\vec{B}}{dt} \, 0$$

$$= \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}$$

2) $\frac{d}{dt}(\phi\vec{A}) = \frac{d\phi}{dt}\vec{A} + \phi\frac{d\vec{A}}{dt}$

Proof: Let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

$$\phi\vec{A} = (\phi A_1)\hat{i} + (\phi A_2)\hat{j} + (\phi A_3)\hat{k}$$

$$\frac{d}{dt}(\phi\vec{A}) = \frac{d}{dt}(\phi A_1)\hat{i} + \frac{d}{dt}(\phi A_2)\hat{j} + \frac{d}{dt}(\phi A_3)\hat{k}$$

$$= \frac{d\phi}{dt}A_1 i + \phi\frac{dA_1}{dt}\hat{i} + \frac{d\phi}{dt}A_2\hat{j} + \phi\frac{dA_2}{dt}\hat{j} + \frac{d\phi}{dt}A_3\hat{k} + \phi\frac{dA_3}{dt}\hat{k}$$

$$= \frac{d\phi}{dt}(A_1\hat{i} + A_2\hat{j} + A_3\hat{k}) + \phi\left(\frac{dA_1}{dt}i + \frac{dA_2}{dt}\hat{j} + \frac{dA_3}{dt}\hat{k}\right)$$

$$= \frac{d\phi}{dt}\vec{A} + \phi\frac{d\vec{A}}{dt}$$

① Ex: Show that $\frac{d}{dt}(\vec{A}.\vec{A}) = 2\vec{A}.\frac{d\vec{A}}{dt}$

② Ex: Show that If $|\vec{A}|$ = constant then $\vec{A} \perp \frac{d\vec{A}}{dt}$ if $|\frac{d\vec{A}}{dt}| \neq 0$

Ex Show that $\vec{A}.\frac{d\vec{A}}{dt} = A\frac{dA}{dt}$

Ex: Let $\vec{A} = t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}$
and $\vec{B} = (2t-3)\hat{i} + \hat{j} - t\hat{k}$
find
i) $\frac{d}{dt}(\vec{A}.\vec{B})$ (ii) $\frac{d}{dt}(\vec{A}\times\vec{B})$

Ex ~~If $\phi(x,y,z) = xyz$~~